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Convergence of Block Spins Defined by a Random Field

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We study the asymptotic behavior of families of dependent random variables called "block spins," which are associated with random fields arising in statistical mechanics. We give sufficient conditions for these families to converge weakly to products of independent Gaussian random variables. We also estimate the error terms involved. In addition we give some conditions which imply that the block spins can converge weakly only to families of normal or degenerate random variables. Central to our proofs is a mixing property which is weaker than strong mixing and which holds for many random fields studied in statistical mechanics. Finally we give a simple method for determining when a stationary random field does not satisfy a strong mixing property. This method implies that the two-dimensional Ising model at the critical temperature is not strong mixing, a result obtained by a different method by M. Cassandro and G. Jona-Lasinio. The method also shows that a stationary, mean-zero, positively correlated Gaussian process indexed by $\mathbb{R}$ is not strong mixing if its covariance function decreases like $t^{-\alpha}$, $0 < \alpha < 1$.

KEY WORDS: Block spins; random field; mixing random variables; Ising model.

1. INTRODUCTION

Let Z^d, $d \geqslant 1$, denote the integer lattice points in d-dimensional Euclidean space. Throughout this paper $d(\cdot, \cdot)$ will denote Euclidean distance in Z^d and $\|\cdot\|$ will denote cardinality.

Let $(X(\mathbf{n}))_{\mathbf{n}\in Z^d}$ be a random field, i.e., an array of random variables indexed by Z^d and defined on some probability space $(\Omega, \mathscr{F}, P)$. We will assume here that $\Omega = \mathbb{R}^d$, $\mathscr{F}$ is the σ-algebra generated by finite-dimensional cylinder sets, and, for $\omega \in \Omega$ and $\mathbf{n} \in Z^d$, $X(\mathbf{n})(\omega) = X(\mathbf{n}, \omega) = \omega(\mathbf{n})$. For a given positive integer N we wish also to consider the reduced lattice obtained

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by breaking Z^d into disjoint d-dimensional cubes of side N and relabeling these blocks in a natural way. Thus for $\mathbf{n} = (n_1, \ldots, n_d) \in Z^d$ we define the block $B^N(\mathbf{n})$ by

$$B^N(\mathbf{n}) = \{\mathbf{m} \in Z^d | Nn_i \leqslant m_i < N(n_i + 1), \quad 1 \leqslant i \leqslant d\} \tag{1.1}$$

Clearly, as $N \to \infty$, $\|B^N(\mathbf{n})\| \sim N^d$ for each $\mathbf{n}$. We now define for each N a new family of random variables (or a new random field) $(X^N(\mathbf{n}))_{\mathbf{n} \in Z^d}$ by setting

$$X^N(\mathbf{n}) = \sum_{\mathbf{m} \in B^N(\mathbf{n})} X(\mathbf{m}), \qquad \mathbf{n} \in Z^d \tag{1.2}$$

The random variables defined by (1.2) will be called the (*Nth-step*) *block spins.*

The study of the family of block spins is motivated by examples from statistical mechanics, notably the Ising model.[4–6,12] The possible limit distributions of these families are thought of as "fixed points" for certain renormalization transformations,[12] and it is conjectured that these limit distributions must be normal above the critical temperature but that nonnormal limits may occur at the critical temperature. This behavior is a result of increasing dependence among the $X(\mathbf{n})$ as the critical temperature is approached, where the dependence is best described as a type of "mixing."

Definition 1. For $A \subset Z^d$ let $\mathscr{F}_A$ denote the sub-σ-algebra of $\mathscr{F}$ generated by the cylinder sets over A. We say the measure P [or the associated family $X(\mathbf{n})$] *satisfies a strong mixing condition* if there exists a function α: $[1, \infty) \to (0, \infty)$, $\alpha(t) \downarrow 0$ as $t \to \infty$, such that whenever $A, B \subset Z^d$ with $d(A, B) = t$, $E \in \mathscr{F}_A$, and $F \in \mathscr{F}_B$,

$$(\mathrm{M_s}) \qquad |P(EF) - P(E)P(F)| \leqslant \alpha(t)$$

We say that P [or the family of $X(\mathbf{n})$] *satisfies a mixing condition* if the previous statement holds with $(\mathrm{M_s})$ replaced by

$$(\mathrm{M}) \qquad |P(EF) - P(E)P(F)| \leqslant \alpha(t)\|A\|$$

In Ref. 8 it is shown that for a random field satisfying a mixing condition similar to $(\mathrm{M_s})$, with A and B parallel hyperplanes, the limits of the block spins, if they exist, must be independent and normally distributed. In Ref. 6 some sufficient conditions for convergence are given and the behavior of the block spins at the critical temperature for $d = 2$ is determined under some very strong conjectures. No estimates of error terms are given in these papers. Behavior at the critical temperature, including the failure of $(\mathrm{M_s})$ in $d = 2$, is also discussed in Ref. 1. In this paper we study the block spins of random fields satisfying (M), a condition introduced in Refs. 3 and 9. Using (M) enables us to make estimates about the error terms involved, for example, since we can consider more varied types of subsets than the hyperplanes considered in the mixing condition used in Ref. 8. The criterion for determining when (M) is satisfied (Theorems 2 and 5, Ref. 3) is also relatively easy to verify for a wide range of models.

In Section 2 we show that if (M) holds with $\alpha(t) \downarrow 0$ rapidly enough, then the block spins must converge weakly to the product of independent Gaussian random variables. In Section 3, for a stationary random field, we find a sufficient rate of decay for (M) to imply that the weak limits of the block spins, if they exist, must be normal or degenerate. Finally in Section 4 we use the exact calculation of the correlation function of the two-dimensional Ising model at the critical temperature,[14,15] together with a lemma from Ref. 10, to show that property (M_s) does not hold for this model. (Another proof of this result may be found in Ref. 1.) We also give some one-dimensional examples in which (M_s) fails, even though covariance terms $E[X(0) - E(X(0))] \times [X(n) - E(X(n))]$ decrease to zero as $n \to \infty$.

2. CONVERGENCE OF THE BLOCK SPINS

We need the following fundamental lemma, the proof of which may be found in Ref. 2:

Lemma 2.1. Suppose (M) holds. Let $A, B \subset Z^d$ with $d(A, B) = t$, $f \in \mathscr{F}_A$, $g \in \mathscr{F}_B$, and $\|f\|_p < \infty$, $\|g\|_q < \infty$, where the norms are with respect to the measure P. If $p, q, r \geqslant 1$ and $1/p + 1/q + 1/r = 1$, then

$$|E(fg) - E(f)E(g)| \leqslant [4\alpha(t)\|A\|]^{1/r}\|f\|_p\|g\|_q$$

If $p = q = \infty$, then

$$|E(fg) - E(f)E(g)| \leqslant 4\alpha(t)\|A\| \, \|f\|_\infty\|g\|_\infty$$

Now, for $N \geqslant 1$ and $\mathbf{n} \in Z^d$ let $X^N(\mathbf{n})$ be defined by (1.2). Then $(X^N(\cdot))_{N\geq 1}$ is a sequence of random functions on the space Z^d and, following Ref. 7, we say the X^N converge weakly to a random function $(Y(\mathbf{n}))_{\mathbf{n}\in Z^d}$ if for each $p \geqslant 1$ and $\mathbf{n}_1,\ldots, \mathbf{n}_p \in Z^d$ the joint distribution function of $(X^N(\mathbf{n}_1),\ldots, X^N(\mathbf{n}_p))$ converges to the joint distribution function of $(Y(\mathbf{n}_1),\ldots, Y(\mathbf{n}_p))$. We assume throughout this section that $E(X(\mathbf{n})) = 0$ for $\mathbf{n} \in Z^d$. We assume also that either

$$\begin{aligned}&\|X(\mathbf{n})\|_\infty \leqslant C < \infty \quad \text{for all } \mathbf{n} \in Z^d\\ &\text{and (M) is satisfied with} \quad \alpha(t) \ll t^{-d-\nu} \text{ for some } \nu > 0\end{aligned} \tag{2.1}$$

or

$$\begin{aligned}&E|X(\mathbf{n})|^{2+\delta} \leqslant C < \infty \quad \text{for some } 0 < \delta < 1 \text{ and all } \mathbf{n} \in Z^d\\ &\text{and (M) is satisfied with} \quad \int_1^\infty \alpha^{\delta/(2+\delta)}(t)t^{d-1}\,dt < \infty\end{aligned} \tag{2.2}$$

Clearly Lemma 2.1 and either (2.1) or (2.2) imply that for $A \subset Z^d$

$$E\left|\sum_{\mathbf{n}\in A} X(\mathbf{n})\right|^2 \ll \|A\| \tag{2.3}$$

Now for each $N \geqslant 1$ and $\mathbf{n} \in Z^d$ let $\partial B^N(\mathbf{n})$ denote the usual topological boundary of $B^N(\mathbf{n})$ and for $0 < \epsilon < 1$ set

$$\partial_\epsilon B^N(\mathbf{n}) = \{\mathbf{m} \in B^N(\mathbf{n}) | d(\{\mathbf{m}\}, \partial B^N(\mathbf{n})) \leqslant N^\epsilon\}$$

so that

$$\|\partial_\epsilon B^N(\mathbf{n})\| \ll N^{d-1+\epsilon} \tag{2.4}$$

Lemma 2.2. Suppose $(X(\mathbf{n}))_{\mathbf{n}\in Z^d}$ satisfies the above conditions. Suppose there exist constants $(C^N(\mathbf{n}))_{\mathbf{n}\in Z^d, N\geq 1}$, such that for each $N \geqslant 1$ and $\mathbf{n} \in Z^d$

$$C^N(\mathbf{n}) \gg N^{d/2} \tag{2.5}$$

and the sequence $X^N(\mathbf{n})/C^N(\mathbf{n})$ converges weakly to a random variable $\tilde{Y}(\mathbf{n})$. Then the sequence of random functions $(\tilde{X}^N(\mathbf{n}))_{\mathbf{n}\in Z^d}$ defined by $\tilde{X}^N(\mathbf{n}) = X^N(\mathbf{n})/C^N(\mathbf{n})$ converges weakly to the random function $(Y(\mathbf{n}))_{\mathbf{n}\in Z^d}$ where, for $\mathbf{n} \in Z^d$, $Y(\mathbf{n})$ has the same distribution function as $\tilde{Y}(\mathbf{n})$ and the $Y(\mathbf{n})$ are independent.

Proof. For $\mathbf{n} \in Z^d$ and $t \in \mathbb{R}$ let $g_{\mathbf{n}}(t)$ be the characteristic function of $\tilde{Y}(\mathbf{n})$. Then it suffices to show that if $p \geqslant 1$, $\mathbf{n}_1,\ldots, \mathbf{n}_p \in Z^d$, and $t_1,\ldots, t_p \in \mathbb{R}$,

$$E\left\{\exp i\left[\sum_{1\leq j\leq p} \frac{t_j X^N(\mathbf{n}_j)}{C^N(\mathbf{n}_j)}\right]\right\} \to \prod_{1\leq j\leq p} g_{\mathbf{n}_j}(t_j) \tag{2.6}$$

The standard method of proving the corresponding result when the $X(\mathbf{n})$ are independent and the $\tilde{Y}(\mathbf{n})$ are normal, for example, is to rearrange the summands and then to appeal to the one-dimensional central limit theorem (Ref. 7, p. 19), but in our case we can make better use of the mixing assumption if we proceed more directly. Let $0 < \epsilon < 1$ and

$$Z^N(\mathbf{n}_j) = \sum_{\mathbf{n}\in\partial_\epsilon B^N(\mathbf{n}_j)} \frac{X(\mathbf{n})}{C^N(\mathbf{n}_j)}, \qquad W^N(\mathbf{n}_j) = \frac{X^N(\mathbf{n}_j)}{C^N(\mathbf{n}_j)} - Z^N(\mathbf{n}_j) \tag{2.7}$$

Now, by (2.3)–(2.5),

$$E(Z^N(\mathbf{n}_j))^2 = o(1) \qquad \text{as } N \to \infty$$

and thus by a modification of the standard proof for $p = 1$ we have

$$E\left\{\exp i\left[\sum_{1\leq j\leq p} \frac{t_j X^N(\mathbf{n}_j)}{C^N(\mathbf{n}_j)}\right]\right\} = E\left\{\exp i\left[\sum_{1\leq j\leq p} t_j W^N(\mathbf{n}_j)\right]\right\} + o(1) \tag{2.8}$$

and a fortiori

$$E\left[\exp \frac{it_j X^N(\mathbf{n}_j)}{C^N(\mathbf{n}_j)}\right] = E[\exp it_j W^N(\mathbf{n}_j)] + o(1) \tag{2.9}$$

Finally, since for any $j \neq j'$ the sites involved in the definition of $W^N(\mathbf{n}_j)$ are all at least distance $2N^\epsilon$ away from any site involved in the definition of $W^N(\mathbf{n}_{j'})$ and since by (2.1) or (2.2), if we choose ϵ close enough to 1, $\alpha(N^\epsilon)N^d = o(1)$ as $N \to \infty$, we can apply Lemma 2.1 a total of $p - 1$ times to give

$$E\left\{\exp i\left[\sum_{1 \leqslant j \leqslant p} \frac{t_j W^N(\mathbf{n}_j)}{C^N(\mathbf{n}_j)}\right]\right\} - \prod_{1 \leqslant j \leqslant p} E\left[\exp \frac{it_j W^N(\mathbf{n}_j)}{C^N(\mathbf{n}_j)}\right] = o(1) \quad (2.10)$$

Now clearly (2.8)–(2.10) imply (2.6).

Letting $(\Phi(\mathbf{n}))_{\mathbf{n} \in Z^d}$ denote an independent family of normal random variables with $E(\Phi(\mathbf{n})) = 0$ and $E(\Phi(\mathbf{n}))^2 = 1$ for each $\mathbf{n} \in Z^d$, we can show the following result:

Theorem 2.3. Suppose $E(X(\mathbf{n})) = 0$ and $E|X(\mathbf{n})|^5 \leqslant C < \infty$ for $\mathbf{n} \in Z^d$, (M) is satisfied with

$$\int_1^\infty \alpha^{1/5}(t) t^{2d-1}\, dt < \infty \quad (2.11)$$

and there exist C and $R > 0$ such that if $A \subseteq Z^d$ and $\|A\| \geqslant R$,

$$E\left[\sum_{n \in A} X(\mathbf{n})\right]^2 \geqslant C\|A\| \quad (2.12)$$

Set

$$\Sigma_{N,\mathbf{n}}^2 = E[X^N(\mathbf{n})]^2 \quad (2.13)$$

Then the sequence of random functions

$$(X^N(\mathbf{n})/\Sigma_{N,\mathbf{n}})_{\mathbf{n} \in Z^d, N \geqslant 1}$$

converges weakly to

$$(\Phi_\mathbf{n})_{\mathbf{n} \in Z^d}$$

Proof. By a slight modification of the proof of Theorem (3.3) in Ref. 16 we can show that for each $\mathbf{n}$, $X^N(\mathbf{n})/\Sigma_{N,\mathbf{n}}$ converges weakly to $\Phi(\mathbf{n})$. Since (2.12) implies (2.5), we can then apply Lemma 2.2.

Let us note that by Lemma 2.1 and (2.1) or (2.2), condition (2.12) is satisfied, for example, if the $X(\mathbf{n})$ are stationary with $E(X(\mathbf{k})X(\mathbf{n})) \geqslant 0$ for $\mathbf{k}, \mathbf{n} \in Z^d$ and $|E(X(\mathbf{k})X(\mathbf{n}))| \ll \alpha(d(\{\mathbf{k}\}, \{\mathbf{n}\}))$ for $\alpha(\cdot)$ as in (2.1) or (2.2). This is the case, for example, for the Ising model above the critical temperature if the interaction potential is assumed to be positive, since we may take $\alpha(t) = e^{-\beta t}$ for some $\beta > 0$ (see Ref. 8 and the discussion in Ref. 16, Section 5, for example).

For uniformly bounded $X(\mathbf{n})$ we can do slightly better:

Theorem 2.4. Suppose $E(X(\mathbf{n})) = 0$ and $\|X(\mathbf{n})\|_\infty \leqslant C < \infty$ for $\mathbf{n} \in Z^d$. Suppose (M) is satisfied with $\alpha(t) \ll t^{-2d-\epsilon} \wedge t^{-3}$ for some $\epsilon > 0$ and suppose (2.12) holds. Then the conclusion of Theorem 2.3 is valid.

Proof. We need only combine the conclusion of Theorem (3.4) of Ref. 17 with our Lemma 2.2.

Results similar to Theorems 2.3 and 2.4 are contained in Ref. 6, but no estimates of error terms are given there. Here we do obtain some estimates. Note that for the rectangle sets which we consider the error terms we find are essentially independent of the relative positions of the sites $\mathbf{n}_1, \ldots, \mathbf{n}_p$.

Theorem 2.5. Suppose $E(X(\mathbf{n})) = 0$ and $E|X(\mathbf{n})|^5 \leqslant C < \infty$ for $\mathbf{n} \in Z^d$, $\Sigma_{N,\mathbf{n}}$ is defined by (2.13), and (2.12) holds. Suppose (M) is satisfied with $\alpha(t) = e^{-\beta t}$ for some $\beta > 0$. Let $\mathbf{n}_1, \ldots, \mathbf{n}_p \in Z^d$ and set $\delta = \min_{1 \leqslant j < k \leqslant p} d(\{\mathbf{n}_j\}, \{\mathbf{n}_k\})$. If for each $\mathbf{n} \in Z^d$ we let $F_\mathbf{n}$ be the distribution function of a normal, mean-0, variance-1, random variable, then for each $x_1, \ldots, x_p \in \mathbb{R}$, as $N \to \infty$,

$$P(X^N(\mathbf{n}_j)/\Sigma_{N,\mathbf{n}_j} < x_j, \quad 1 \leqslant j \leqslant p) - F_{\mathbf{n}_1}(x_1) \cdots F_{\mathbf{n}_p}(x_p) = O\begin{cases} N^{-3/10} & \text{if } d > 1 \\ N^{-1/4} & \text{if } d = 1 \end{cases} \tag{2.14}$$

Proof. By Theorem (3.1) of Ref. 17 the result is true for $p = 1$. Now for $1 \leqslant j \leqslant p$ let $Z^N(\mathbf{n}_j)$ and $W^N(\mathbf{n}_j)$ be defined by (2.7), where $C^N(\mathbf{n}_j) = \Sigma_{N,\mathbf{n}_j}$ and $\epsilon = \epsilon(N)$ will be chosen below. Then for $\eta(N) > 0$, which will also be chosen below, and for $x_1, \ldots, x_p \in \mathbb{R}$,

$$\begin{aligned} &P(X^N(\mathbf{n}_j)/\Sigma_{N,\mathbf{n}_j} < x_j, \quad 1 \leqslant j \leqslant p) \\ &= P\left(X^N(\mathbf{n}_j)/\Sigma_{N,\mathbf{n}_j} < x_j, \quad 1 \leqslant j \leqslant p, \quad \max_{1 \leqslant j \leqslant p} |Z^N(\mathbf{n}_j)| < \eta(N)\right) \\ &\quad + \sum_{1 \leqslant j \leqslant p} P(X^N(\mathbf{n}_j)/\Sigma_{N,\mathbf{n}_j} < x_j, \quad 1 \leqslant j \leqslant p, \\ &\qquad\qquad |Z^N(\mathbf{n}_i)| < \eta(N), \quad 1 \leqslant i < j, \quad |Z^N(\mathbf{n}_j)| \geqslant \eta(N)) \\ &= P(W^N(\mathbf{n}_j) < x_j + \eta(N), \quad 1 \leqslant j \leqslant p) \\ &\quad + O\left(\sum_{1 \leqslant j \leqslant p} \frac{E[Z^N(\mathbf{n}_j)]^2}{[\eta(N)]^2}\right) \end{aligned} \tag{2.15}$$

Now if we set $\tilde{B}^N(\mathbf{n}_j) = B^N(\mathbf{n}_j) - \partial_{\epsilon(N)} B^N(\mathbf{n}_j)$, $W^N(\mathbf{n}_j) \in \mathscr{F}_{\tilde{B}^N}(\mathbf{n}_j)$, $1 \leqslant j \leqslant p$, and therefore applying (M) a total of $p - 1$ times gives

$$\begin{aligned} &P(W^N(\mathbf{n}_j) < x_j + \eta(N), \quad 1 \leqslant j \leqslant p) \\ &\quad = \prod_{1 \leqslant j \leqslant p} P(W^N(\mathbf{n}_j) < x_j + \eta(N)) + O(pN^d \alpha(2\delta N^\epsilon)) \end{aligned} \tag{2.16}$$

Now if we set $\tilde{\Sigma}^2_{N,\mathbf{n}_j} = E[\Sigma_{N,\mathbf{n}_j} W^N(\mathbf{n}_j)]^2$, then $\Sigma_{N,\mathbf{n}_j} = \tilde{\Sigma}_{N,\mathbf{n}_j}[1 + o(1)]$. This together with Theorem (3.1) of Ref. 17 implies

$$\begin{aligned} &P(W^N(\mathbf{n}_j) < x_j + \eta(N)) \\ &\quad = F_{\mathbf{n}_j}(x_j + \eta(N)) + O\begin{pmatrix} N^{-3/10}[1 + o(1)], & d > 1 \\ N^{-1/4}[1 + o(1)], & d = 1 \end{pmatrix} \end{aligned} \tag{2.17}$$

Now setting $\eta(N) = N^{-3/10}$ and $\epsilon(N) = 1/10$ and combining (2.15)–(2.17) gives the desired result.

We can obtain results analogous to Theorem 2.5 but with a larger error term if we relax the requirement that the mixing be exponential. For example, if the hypotheses of Theorem 2.5 hold with $\alpha(t)$ satisfying (2.11) or if the hypotheses of Theorem 2.4 hold we can use Theorem (3.2) or Theorem (3.4) of Ref. 17 to obtain an estimate like (2.16) and thus a remainder term of $N^{-1/8}$. Estimate (2.14) can, of course, be used to approximate the distribution function of

$$\max_{1 \leqslant j \leqslant p} [X^N(\mathbf{n}_j)/\Sigma_{N,\mathbf{n}_j}]$$

Other results in Ref. 17 can also be applied to the block spins, for example

Proposition 2.6. For $\mathbf{n}_1,\ldots,\mathbf{n}_p \in Z^d$ and $N \geqslant 1$ let $t^N(\mathbf{n}_j) = 2\Sigma^2_{N,\mathbf{n}_j} \times \log(\log \Sigma^2_{N,\mathbf{n}_j})$. Suppose the hypotheses of Theorem 2.5 hold and $(a^N(1),\ldots, a^N(p))$ is a sequence of vectors with nonzero entries such that for $1 \leqslant j \leqslant p$, $(t^N(\mathbf{n}_j))^{1/2} = o(a^N(j))$ as $N \to \infty$. Then as $N \to \infty$ the sequence $(X^N(\mathbf{n}_1)/a^N(1),\ldots, X^N(\mathbf{n}_p)/a^N(p))$ converges a.s. to the zero vector.

Proof. For $p = 1$ this is just Corollary (3.10) of Ref. 17. Now for $1 \leqslant j \leqslant p$ let

$$\Lambda_j = \{\omega: \quad X^N(\mathbf{n}_j)/a^N(j)) \nrightarrow 0\}$$

Clearly $P(\bigcup_{1 \leqslant j \leqslant p} \Lambda_j) = 0$ and, for $\omega \in (\bigcup_{1 \leqslant j \leqslant p} \Lambda_j)^c$, $X^N(\mathbf{n}_j, \omega)/a^N(j) \to 0$ for $1 \leqslant j \leqslant p$.

3. CONVERGENCE TO A STABLE LAW

Here we determine conditions on the mixing function α which imply that one block spin, if it converges weakly, must converge to a normal law. From Ref. 11, Theorem (18.1.1), it follows that for $d = 1$ and $(X(\mathbf{n}))_{\mathbf{n} \in Z}$ identically distributed, $(\mathrm{M_s})$ being satisfied with any $\alpha(t) \to 0$ is sufficient to imply that sums $(X(1) + \cdots + X(N))/B_N - A_N$, $B_N \to \infty$, can only converge weakly to a stable or a degenerate distribution. Furthermore, if the limit distribution is stable with exponent α, then $B_N = N^{1/\alpha}h(N)$, where $h(N)$ is slowly varying as $N \to \infty$. Similar results are obtained in Ref. 8 for random fields satisfying conditions like $(\mathrm{M_s})$, but it is necessary to assume that certain "boundary terms" also behave properly. The conditions we impose on the function $\alpha(\cdot)$ and on the moments of the $X(\mathbf{n})$ have the effect of ensuring that these boundary terms do behave properly. Working with (M) instead of $(\mathrm{M_s})$ means the conditions on $\alpha(\cdot)$ must be even more stringent. Our Theorem 3.2 below deals with only one block spin. We point out in Corollary 3.3 that it can be combined with Lemma 2.2 to give information about the joint distributions of any (finite) number of block spins. The replacing of the blocks $B^N(\mathbf{0})$ by d-dimensional rectangles in Theorem 3.2 may seem somewhat unnatural at first glance, but it is unavoidable, since rectangles, not cubes, are the fundamental building blocks for measurable sets if $d > 1$. The use of rectangles instead of cubes is not really much of a change. If we consider the case of convergence to a normal law, for example, it is clear that the proofs in Section 2 of this paper and in Ref. 17 can easily be made valid for any sequence of d-dimensional rectangles $\uparrow Z^d$ in such a way that the rates of growth in any two directions are proportional. The proper normalizing constant in these cases is determined by the fact that for a rectangle R

$$E\left[\sum_{\mathbf{n}\in R} X(\mathbf{n})\right]^2 \sim \|R\|$$

Definition 3.1. The array $(X(\mathbf{n}))_{\mathbf{n}\in Z^d}$ will be called *strictly stationary* if for all $p \geqslant 1$ and $\mathbf{n}_1,\ldots,\mathbf{n}_p$, $n \in Z^d$, both $(X(\mathbf{n}_1),\ldots,X(\mathbf{n}_p))$ and $(X(\mathbf{n}_1 + \mathbf{n}),\ldots, X(\mathbf{n}_p + \mathbf{n}))$ have the same joint distribution functions.

Theorem 3.2. Suppose $(X(\mathbf{n}))_{\mathbf{n}\in Z^d}$ is strictly stationary with $E|X(\mathbf{0})|^{2+\delta} < \infty$ for some $\delta > 0$. Suppose (M) is satisfied with

$$\alpha(t) \ll t^{-d/v} \tag{3.1}$$

for some $0 < v < 1$.

Suppose there exists a function f: $\{1, 2, 3,\ldots\} \to \mathbb{R}$ with $f\uparrow$ and

$$f(N^d) \gg N^{(d-1+v')/2} \tag{3.2}$$

where $v < v' < 1$. Suppose also that there is a nondegenerate distribution function $F(x)$ such that for any sequence of d-dimensional rectangles $R_N \uparrow Z^d$ there is a sequence $(A(R_N))$ with $F_{R^N}(x)$, the distribution functions of

$$\left[\sum_{\mathbf{n}\in R^N} X(\mathbf{n})\right] \Big/ f(\|R_N\|) - A(R_N)$$

converging weakly to $F(x)$. Then $F(x)$ is normal. Furthermore, if $B^N(\mathbf{0})$ is defined as in (1.1),

$$f(\|B^N(\mathbf{0})\|) = \|B^N(\mathbf{0})\|^{1/2} h(\|B^N(\mathbf{0})\|) \tag{3.3}$$

where $h(y)$ is slowly varying as $y \to \infty$.

Proof. Let $B^N = B^N(\mathbf{0})$ as in (1.1). For any rectangle R let $S_R = \sum_{n\in R} X(n)$ and let F_R be the distribution function of S_R. Choose v'',

$$v < v'' < v' \tag{3.4}$$

and let C^N be the rectangle obtained by adding to B^N those sites $\mathbf{n} \in Z^d$ with

$$N \leqslant n_1 < [N + N^{v''}], \qquad 0 \leqslant n_i < N, \quad 2 \leqslant i \leqslant d$$

Let D^M be obtained by translating B^M so that $[N + N^{v''}] \leqslant n_1 < [N + N^{v''} + M]$. Let $f_N = f(\|B^N\|)$ and $f_M = f(\|B^M\|)$. Now by Lemma 2.1 with $p = q = 2 + \delta$

$$E|S_{C^N} - S_{B^N}|^2 \ll N^{d-1+v''}$$

and thus

$$f_N^{-1}(S_{C^N} - S_{B^N}) \to 0 \text{ in probability as } N \to \infty$$

Now as in the proof for $d = 1$ (p. 316, Ref. 11), we can show that for any $a_1, a_2 > 0$ there is a sequence $M(N) \to \infty$ with

$$\lim_{N\to\infty} f_{M(N)}/f_N = a_1/a_2$$

For $b_1, b_2 \in \mathbb{R}$ set

$$Y_N = a_1^{-1}(f_N^{-1}S_{B^N} - A_{B^N} - b_1)$$

and

$$\tilde{Y}_N = (f_{M(N)}/a_1 f_N)(f_{M(N)}^{-1}(S_{C^N \cup D^{M(N)}} - S_{C^N} - A_{M(N)} - b_2))$$

Clearly

$$Y_N + \tilde{Y}_N = ((a_1 f_N)^{-1} S_{C^N \cup D^{M(N)}} - A_N') - (a_1 f_N)^{-1}(S_{C^N} - S_{B^N})$$

and

$$Y_N \in \mathscr{F}_{B^N}, \qquad \tilde{Y}_N \in \mathscr{F}_{D^{M(N)}}$$

Thus by (M) and stationarity the distribution function of $Y_N + \tilde{Y}_N$ differs from

$$F_{B^N}(a_1 x + b_1) * F_{D^{M(N)}}((a_1 f_N/f_{M(N)})x + b_2)$$

by at most $\|B^N\|\alpha(N^{\nu''}) = o(1)$. By the definition of C^N the distribution function of $Y_N + \tilde{Y}_N$ approaches $F(ax + b)$ for some constants a and b. Thus $F(x)$ is stable. Now suppose $F(x)$ has exponent α. To show (3.3) we need only show that for any positive integer k

$$\lim_{N\to\infty} f(\|B^{kN}(\mathbf{0})\|)/f(\|B^N(\mathbf{0})\|) = k^{d/\alpha} \tag{3.5}$$

Let ν'' be as in (3.4). Now $B^{kN}(\mathbf{0})$ is obtained by putting together k^d copies of $B^N(\mathbf{0})$. Label these copies $B_{N,1,\dots,}B_{N,k^d}$, and for $1 \leqslant i \leqslant k^d$ let

$$B'_{N,i} = B_{N,i} - \{\mathbf{n} \in Z^d\colon \quad d(\partial B_{N,i}, \{\mathbf{n}\}) \le N^{\nu''}\},$$
$$B''_{N,i} = B_{N,i} - B'_{N,i}$$

and

$$\xi_{N,i} = \sum_{\mathbf{n}\in B_{N,i}} X(\mathbf{n})$$

Now, since by Lemma 2.1 with $p = q = 2 + \delta$

$$E\left|\sum_{\substack{\mathbf{n}\in B''_{N,i}\\ 1\leqslant i\leqslant k-1}} X(\mathbf{n})\right|^2 \ll k^d N^{d-1+\nu''} = o(f_N{}^2)$$

the distribution function of

$$f_{kN}^{-1} \sum_{1 \leqslant j \leqslant k^d} \xi_j - A_{B^{kN}}$$

approaches the same limit as the distribution function of

$${}_{kN}^{-1} \sum_{\mathbf{n} \in B^{kN}(\mathbf{0})} X(\mathbf{n}) - A_{B^{kN}}$$

where $f_{kN} = f(\|B^{kN}\|)$. Furthermore, the ξ_j are identically distributed. By (M),

$$\left| E\left(\exp it \sum_{1 \leqslant j \leqslant k^d} \frac{\xi_j}{f_{kN}}\right) - \prod_{1 \leqslant j \leqslant k^d} E\left(\exp \frac{it\xi_j}{f_{kN}}\right) \right| = O(N^d \alpha(N^{\nu''})) = o(1)$$

and thus for each t, if $\phi_B(\cdot)$ is the characteristic function of $S_B/f_B - A_B$,

$$\left| |\phi_{B^{kN}}(t)| - |\phi_{B^N}(tf_N/f_{kN})|^{k^d} \right| \to 0$$

as $N \to \infty$. But the stability of F implies

$$\lim_{N \to \infty} |\phi_{B^N}(t)| = \exp(-c|t|^\alpha), \qquad c > 0$$

Thus we must have

$$\lim_{N \to \infty} (f_N/f_{kN})^\alpha k^d = 1$$

and so we have shown (3.5). Finally by Lemma 2.1 with $p = q = 2 + \delta$

$$E[X^N(\mathbf{0})]^2 \ll N^d$$

Thus if F has exponent $\alpha < 2$, we must have $E[X^N(\mathbf{0})]^2/f_N^2 \to 0$ as $N \to \infty$ and so F would be degenerate, which is a contradiction.

Corollary 3.3. If $(X(\mathbf{n}))_{\mathbf{n} \in Z^d}$ is strictly stationary with $E|X(\mathbf{0})|^{2+\delta} < \infty$ for some $\delta > 0$, if (3.2) holds, and if the block spins, normalized by constants f_{B^N}, with f as in (3.3), converge weakly to a nondegenerate random function, then the limit distributions must be independent and normal.

Proof. We need only combine the proofs of Theorem 3.2 and Lemma 2.2.

4. SOME EXAMPLES

In this section we give some examples of families which do not satisfy the strong mixing condition (M_s) even though the covariances $E(X(\mathbf{0})X(\mathbf{n}))$ decrease to zero as $d(\{\mathbf{0}\}, \{\mathbf{n}\}) \to \infty$. We start by giving a simple criterion for determining when strong mixing is violated.

Lemma 4.1. Suppose $(X(\mathbf{n}))_{\mathbf{n}\in Z^d}$ is strictly stationary with $E(X(\mathbf{0})) = 0$ and $E|X(\mathbf{0})|^{2+\delta} < \infty$ for some $\delta > 0$. Let $X^N = X^N(\mathbf{0})$ as in (1.2). Suppose there is an $\epsilon > 0$ with

$$E(X^N)^2 \gg N^{d+\epsilon} \tag{4.1}$$

Then the $X(\mathbf{n})$ do not satisfy (M_s).

Proof. We can easily modify the proof of Lemma (1.8) of Ref. 10 to show that if the $X(\mathbf{n})$ are strongly mixing and satisfy all the hypotheses of Lemma 4.1 except for (4.1), then we must have $E(X^N)^2 \sim N^d h(N)$, where $h(N)$ is slowly varying as $N \to \infty$.

Example 4.2. Let $d = 2$ and consider the Ising model with positive nearest neighbor interactions at the critical temperature. By results in Refs. 14 and 15, $E(X(\mathbf{0})X(\mathbf{n})) \sim d(\{\mathbf{0}\}, \{\mathbf{n}\})^{-1/4}$. Thus $E(X^N)^2 \sim N^{11/4}$ and so (4.1) holds with $\epsilon = 3/4$. (Because the interaction potential is assumed positive, $E(X(\mathbf{0})X(\mathbf{n})) \geqslant 0$ for all n. This is discussed in Ref. 16, for example.) A different proof that strong mixing does not hold for this model may be found in Ref. 1.

Example 4.3. Fix α, $0 < \alpha < 1$. Consider a stationary, mean-0 Gaussian sequence $(X(n))_{n\in Z}$ with covariance function

$$\rho(n) = 1/(1 + |n|^{\alpha})$$

[By Polya's condition (Ref. 13, p. 70), ρ is a characteristic function and thus positive definite and therefore there is a Gaussian process with this covariance function.] Clearly in this case

$$E\left[\sum_{n=0}^{N} X(n)\right]^2 \gg N^{2-\alpha}$$

and thus this sequence cannot be strong mixing.

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